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Abstract: We consider two independent Poisson planar processes $\mathcal{C}$ and $\mathcal{S}$ of intensities $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{S}}$ respectively. All the particles of the process $\mathcal{S}$ within the same Voronoi cell of the process $\mathcal{C}$ are connected to its nucleus. We study stochastic characteristics of the number and total length of these connections within a typical Voronoi $\mathcal{C}$-cell. We find first and second moments of these variables as well as exponential asymptotic of the tail decay

Key-words: Bivariate Poisson process, Cluster process, Voronoi tesselation, Palm measures, Rough deviations, Telecommunicational networks

(Résumé : tsvp)

AMS 1991 Subject Classification: 60D05, 52C17, 60G55

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Sur certains processus de segments lies a la tessellation de Voronoi

Résumé : On considère deux processus de Poisson indépendents $\mathcal{C}$ et $\mathcal{S}$ d'intensités $\lambda_{\mathcal{C}}$ et $\lambda_{\mathcal{S}}$ respectivement. Tous les points dans la même cellule de Voronoï par rapport au processus $\mathcal{C}$ sont connectés à son noyau par des segments. Nous donnons les deux premiers moments de la distribution du nombre et de la longueur des ces segments dans une cellule typique, ainsi que l'assymptotique exponentielle du comportement de sa queue.

Introduction

The model we consider in this article has the origins in studies of telecommunicational networks. In some cases the lowest level of such a network can be described as follows.

Suppose we are given two random point processes in a plane $\mathbb{R}^2$: a process $\mathcal{S}$ which members represent 'subscribers' and a process $\mathcal{C}$ representing 'comutators' (or stations). To each subscriber there correspond a station to which this subscriber is attached by a physical link (cable). Quite naturally such a commutator is chosen to be the closest one to the specified subscriber. Thus the plane is devided into regions - cells, each corresponding to a commutator and consisting of those points of $\mathbb{R}^2$ which have the specified commutator as the closest one among all the others. These cells form a tesselation of $\mathbb{R}^2$ known as *a Voronoi tesselation.* For each commutator c_i its Voronoi cell $T(c_i)$ is a convex set sinse it can be represented as an intersection of half-planes: $T(c_i) = \{x \in \mathbb{R}^2 \ : \ |x - c_i| \leq |x - c_j| \text{ for all } j\}$. This c_i is called *a nucleus* of the cell $T(c_i)$.

In this article we suppose that both $\mathcal{C}$ and $\mathcal{S}$ are independent homogeneous Poisson point processes with intensities $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{S}}$ respectively. Due to tradition we use the term *particles* for the members of the point processes.

Let s_i denote the particles (subscribers) of the process $\mathcal{S}$, and $b(c_i, s_j)$ be a segment between particles $c_i \in \mathcal{C}$ and $s_j \in \mathcal{S}$. Then the process under consideration is the process $\mathcal{F} = \{b(c_i, s_j) \ : \ s_j \in T(c_i)\}$ which can be regarded as a process of segments with dependent clustering in germs $c_i \in \mathcal{C}$. We however do not intend to deepen into complete rigorous description of this stochastic process. -1z Our goal is to study geometrical properties of a 'typical' cluster $W(c_i) = \cup_{s_j \in T(c_i)} b(c_i, s_j)$ associated with a particle c_i. In ergodic case an adequate interpretation of the notion 'distribution of a typical cluster' is provided by a Palm distribution i. e. conditional distribution of the structure characteristics of the cluster $W(0) = \cup_{s_j \in T(0)} b(0, s_j)$ given that there is a particle of the ensemble $\mathcal{C}$ in the origin 0 (see *Daley and Vere-Jones* (1988), p. 465). Slivnyak's theorem states that in a Poisson case it suffices just to add an extra particle into the origin 0 in all realisation of a Poisson process $\mathcal{C}$. Corresponding process of centered realizations we will denote further by $\mathcal{C}_0 = \mathcal{C} \cup \{0\}$. Then the distribution of a geometrical structure of the cluster associated with 0 coicides with the Palm distribution of the last (see e. g. *Stoyan et al.* (1987), pp. 114-115). A typical realization of the described process is shown on figure 1. The Voronoi tesselation is represented by dashed lines.

Throughout this paper we use the following notations:

- $\mathbf{P}\{\cdot\}$ stands for a Palm distribution (given there is a commutator in the origin 0);
- $\mathbf{E}\{\cdot\}$ denotes expectation with respect to $\mathbf{P}$;

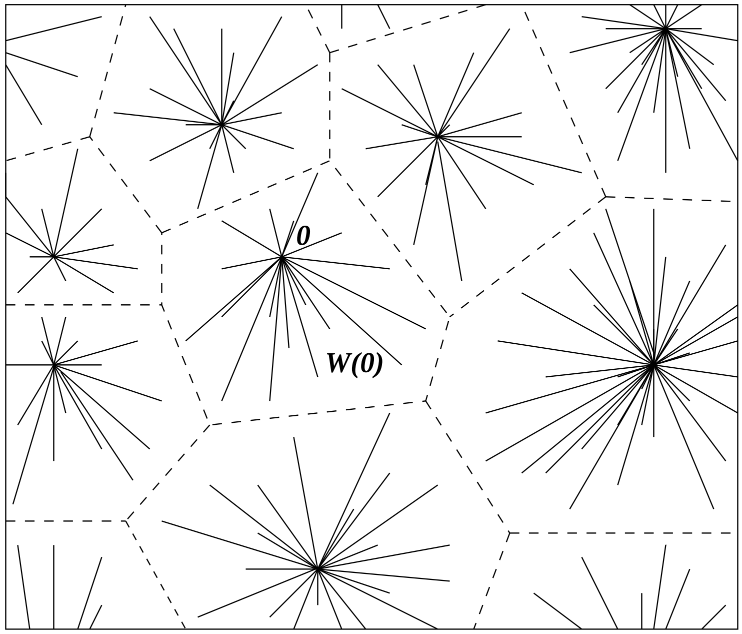

Figure 1

- N is a random variable which is equal to the number of segments in the cluster $W(0)$ associated with 0 (= the number of subscribers in $T(0)$);
- l is the total length of the segments of $W(0)$.

1 First and second moments

Theorem 1 *Moments of the variables N and l are given by formulae:*

$$(i) \quad \mathbf{E}N = \frac{\lambda_{\mathcal{S}}}{\lambda_{\mathcal{C}}}$$

$$(ii) \quad \mathbf{E}l = \frac{\lambda_{\mathcal{S}}}{2\lambda_{\mathcal{C}}^{3/2}}$$

$$(iii) \quad \mathbf{E}N^2 = \frac{\lambda_{\mathcal{S}}}{\lambda_{\mathcal{C}}} + 1.280\frac{\lambda_{\mathcal{S}}^2}{\lambda_{\mathcal{C}}^2};$$

$$\mathbf{var}N = \frac{\lambda_{\mathcal{S}}}{\lambda_{\mathcal{C}}} + 0.280\frac{\lambda_{\mathcal{S}}^2}{\lambda_{\mathcal{C}}^2}$$

$$(iv) \quad \mathbf{E}l^2 = \frac{\lambda_{\mathcal{S}}}{\pi\lambda_{\mathcal{C}}^2} + 0.397\frac{\lambda_{\mathcal{S}}^2}{\lambda_{\mathcal{C}}^3};$$

$$\mathbf{var}\, l = \frac{\lambda_{\mathcal{S}}}{\pi\lambda_{\mathcal{C}}^2} + 0.147\frac{\lambda_{\mathcal{S}}^2}{\lambda_{\mathcal{C}}^3}$$

Proof. (i) The number of segments of $W(0)$ is equal to

$$N = \sum_{i=1}^{\infty} \mathbf{1}\{s_i \in T(0)\}$$

Denote by $\sigma(\mathcal{C}_0)$ – σ-algebra generated by the process $\mathcal{C}_0$. Then

$$\mathbf{E}N = \mathbf{E}\,\mathbf{E}\{N \bigm| \sigma(\mathcal{C}_0)\} = \mathbf{E}\,\mathbf{E}\{\sum_{i=1}^{\infty} \mathbf{1}\{s_i \in T(0)\} \bigm| \sigma(\mathcal{C}_0)\}.$$

Applying Campbell theorem (see e. g. *Stoyan et al.* (1987), p. 99) we find that

$$\mathbf{E}\{\sum_{i=1}^{\infty} \mathbf{1}\{s_i \in T(0)\} \bigm| \sigma(\mathcal{C}_0)\} = \lambda_{\mathcal{S}} \int \mathbf{1}\{x \in T(0)\}dx$$

$$= \lambda_{\mathcal{S}}\nu_2(T(0)),$$

where $\nu_2(T(0))$ is 2-dimentional Lebegue measure (area) of the Voronoi cell $T(0)$. (i) now is immediate since the expectation of $\nu_2(T(0))$ is equal to $1/\lambda_{\mathcal{C}}$ as was established in *Meijering* (1957).

(ii) The same method is applicable for the first moment of the variable l. Denote by $|s_i|$ the Euclidean distance from a subscriber s_i to the origin 0. Then

$$\mathbf{E}\, l = \mathbf{E}\,\mathbf{E}\{\sum_{i=1}^{\infty} |s_i|\mathbf{1}\{s_i \in T(0)\} \bigm| \sigma(\mathcal{C}_0)\}$$

$$= \mathbf{E}\lambda_{\mathcal{S}} \int |x|\mathbf{1}\{x \in T(0)\}dx = \lambda_{\mathcal{S}} \int |x|\mathbf{P}\{x \in T(0)\}dx$$

Due to isotropy of the underlying processes the expression under the integral depends only on the distance r between 0 and the integration point. Thus switching to a polar coordinates and integrating with respect to a polar angle we get

$$\mathbf{E}l = 2\pi\lambda_{\mathcal{S}} \int_0^{+\infty} r^2\, \mathbf{P}\{(r,0) \in T(0)\}\, dr$$

Note that the point $(r, 0)$ belongs to $T(0)$ iff a disc of the radius r around $(r, 0)$ does not contain any particle of the process $\mathcal{C}$ inside. Hence $\mathbf{P}\{(r, 0) \in$

$T(0)\} = \exp\{-\lambda_{\mathcal{C}} \pi r^2\}$ and finally

$$\mathbf{E}l = 2\pi\lambda_{\mathcal{S}} \int_0^{+\infty} r^2 \exp\{-\lambda_{\mathcal{C}} \pi r^2\}\, dr = \frac{\lambda_{\mathcal{S}}}{2\lambda_{\mathcal{C}}^{3/2}}$$

(iii) To evaluate the second moment of the variable N we apply the formula

$$\mathbf{E}\{(\sum_{i=1}^{\infty} f(s_i))^k\} = \int \ldots \int f(x_1)\ldots f(x_k) M_k(dx_1 \ldots dx_k)$$

where M_k is the $k-$th moment measure of the process $\mathcal{S}$ (see e. g. *Daley and Vere-Jones* (1988), p. 190) that gives in our case

$$\mathbf{E}\{(\sum_{i=1}^{\infty} \mathbf{1}\{s_i \in T(0)\})^2 \;\big|\; \sigma(\mathcal{C}_0)\} = \lambda_{\mathcal{S}}^2 \int\int \mathbf{1}\{x_1 \in T(0)\}\mathbf{1}\{x_2 \in T(0)\} dx_1 dx_2$$

$$+\lambda_{\mathcal{S}} \int \mathbf{1}\{x \in T(0)\} dx = \lambda_{\mathcal{S}} \nu_2(T(0)) + \lambda_{\mathcal{S}}^2 \nu_2^2(T(0))$$

We have used here an explicit expression for the second moment measure for a Poisson process

$$M_2(A \times B) = \lambda_{\mathcal{S}} \nu_2(A \cap B) + \lambda_{\mathcal{S}}^2 \nu_2(A)\nu_2(B)$$

(see e. g. *Stoyan et al.* (1987), p. 46).

Taking into account that due to *Gilbert* (1962) $\mathbf{var}\, \nu_2(T(0)) = 0.280/\lambda_{\mathcal{C}}^2$ we have formulae (iii)

(iv) Similarly

$$\mathbf{E}\, l^2 = \mathbf{E}\mathbf{E}\{(\sum_i |s_i|\, \mathbf{1}\{s_i \in T(0)\})^2 \;\big|\; \sigma(\mathcal{C}_0)\}$$

$$= \lambda_{\mathcal{S}} \int |x|^2\, \mathbf{P}\{x \in T(0)\} dx + \lambda_{\mathcal{S}}^2 \int\int |x|\, |y|\, \mathbf{P}\{x, y \in T(0)\} dx\, dy$$

The first integral can be evaluated the same way as $\mathbf{E}l$ giving

$$2\pi\lambda_{\mathcal{S}} \int_0^{+\infty} r^3 \exp\{-\lambda_{\mathcal{C}} \pi r^2\}\, dr = \frac{\lambda_{\mathcal{S}}}{\pi \lambda_{\mathcal{C}}^2}$$

The expression under the second integral depends clearly only on distances from the origin 0 to the points $x = (x_1, x_2),\ y = (y_1, y_2)$ and the angle ψ between their radius-vectors. Thus putting

$$\begin{aligned} x_1 &= r_1 \cos\phi; \\ x_2 &= r_1 \sin\phi; \\ y_1 &= r_2 \cos(\phi + \psi); \\ y_2 &= r_2 \sin(\phi + \psi) \end{aligned}$$

and integrating with respect to ϕ first, we get for the second integral the following expression:

$$4\pi\lambda_{\mathcal{S}}^2 \int_0^{+\infty} dr_1 \int_0^{+\infty} dr_2 \int_0^{\pi} r_1^2\, r_2^2\, \mathbf{P}\{(r_1, 0),\, (r_2\cos\psi,\ r_2\sin\psi) \in T(0)\} d\psi$$

$$= 4\pi\lambda_{\mathcal{S}}^2 I$$

Both points $P_1 = (r_1, 0)$ and $P_2 = (r_2\cos\psi,\ r_2\sin\psi)$ belong to $T(0)$ iff there is no particle of $\mathcal{C}$ in the union of two discs of radii r_1 and r_2 around the points P_1 and P_2 respectively. By elementary geometrical considerations one can find, that the area of this union is given by the formula: $S(r_1, r_2, \psi) = r_1 r_2 \sin\psi + r_1^2\beta_1 + r_2^2\beta_2$, where β_1 and β_2 are the external angles of the triangle $0P_1P_2$ at the vertices P_1 and P_2 respectively (see fig. 2).

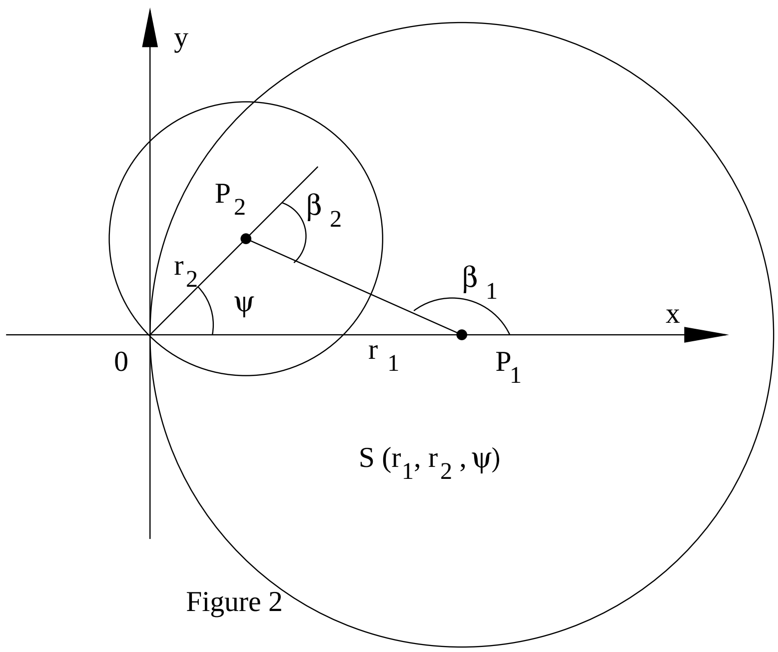

Figure 2

To evaluate the integral

$$I = \int_0^{+\infty} dr_1 \int_0^{+\infty} dr_2 \int_0^{\pi} r_1^2\, r_2^2\, \exp\{-\lambda_{\mathcal{C}} S(r_1, r_2, \psi)\} d\psi$$

we change the integration variables taking as new parameters the angle ψ, the radius R of a circumdisc over the triangle OP_1P_2 and the polar angle α of its circumcenter (see fig. 3).

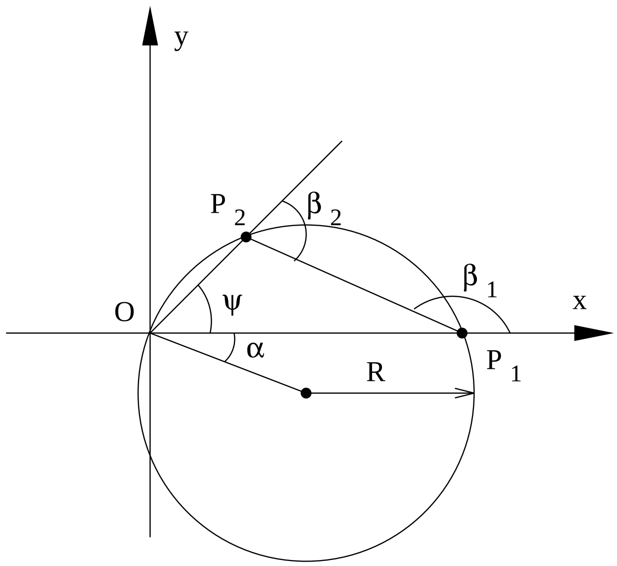

Figure 3

In the new coordinates

$$\begin{aligned} r_1 &= 2R\cos\alpha; \\ r_2 &= 2R\sin(\psi-\alpha); \\ \beta_1 &= \pi/2-\alpha+\psi; \\ \beta_2 &= \pi/2+\alpha; \end{aligned}$$

$$\alpha \in (-\pi/2,\pi/2),\ \psi\in(0,\alpha+\pi/2),\ R\in[0,+\infty).$$

The Jackobian is equal to

$$\left|\frac{D(r_1,r_2)}{D(R,\alpha)}\right| = |4R\sin\psi| = 4R\sin\psi,$$

since $\sin\psi \geq 0$.

Thus we come to the following integral

$$\begin{aligned} I &= 64\int_0^{+\infty} R^5 dR \int_{-\pi/2}^{\pi/2} d\alpha \int_0^{\alpha+\pi/2} \cos^2\alpha\ \cos^2(\psi-\alpha)\ \sin\psi \\ &\quad \times \exp\{-4\lambda_{\mathcal{C}} R^2(\cos\alpha\ \cos(\psi-\alpha)\ \sin\psi + (\pi/2+\psi-\alpha)\cos^2\alpha \\ &\quad +(\pi/2+\alpha)\cos^2(\psi-\alpha)\}\, d\psi, \end{aligned}$$

which can be integrated with respect to R leading to

$$I = \frac{1}{\lambda_C^3} \int_{-\pi/2}^{\pi/2} d\alpha \int_0^{\alpha+\pi/2} \cos^2\alpha\, \cos^2(\psi-\alpha)\, \sin\psi$$
$$\times(\cos\alpha\, \cos(\psi-\alpha)\, \sin\psi + (\pi/2+\psi-\alpha)\cos^2\alpha + (\pi/2+\alpha)\cos^2(\psi-\alpha))^{-3}\, d\psi$$

$$= \frac{1}{\lambda_C^3} \int_0^{\pi} du \int_0^{\pi-u} \frac{\sin^2 u\, \sin^2 v\, \sin(u+v)\, dv}{(\sin u\, \sin v\, \sin(u+v) + (\pi-v)\sin^2 u + (\pi-u)\sin^2 v)^3}$$

for $u = \pi/2 - \alpha$ and $v = \pi/2 - \psi + \alpha$.

Unfortunally there is no way to evaluate this integral but only numerically. Numerical integration also meets some dificulties here because of a pole of the integrand in O. To avoid this difficulty we have used the polar coordinates $u = \rho\cos\chi$; $v = \rho\sin\chi$, where $\chi \in (0, \pi/2)$, $\rho \in (0, \pi(\sin\chi + \cos\chi)^{-1})$. Now multiplied by Jackobian ρ the expression under the integral is bounded in the integration region.

Approximate value of this integral is $0.0316/\lambda_C^3$ finally giving formulae (iv). $\square$

2 Decay of tail probabilities

For practical reasons its important to have not only first moments but also to know how fast the tail of the distribution of the variables N and l falls down. In this section we find an exponential asymptotics for probabilities of large deviations.

The key role in the further considerations play the following two simple lemmas concerning geometrical properties of a Voronoi cell.

Lemma 1 *Let r denote the radius of the maximal ball B_m centered in a nucleus and contained in a Voronoi cell with respect to a Poisson process of intensity λ in $\mathbb{R}^d$. Then*

$$\mathbf{P}\{r > x\} = \exp(-\lambda 2^d r^d K_d) \tag{1}$$

where $K_d = \pi^{k/2}/\Gamma(1+k/2)$ is volume of a unit ball in $\mathbb{R}^d$.

Proof follows easily from the fact that the event $\{r > x\}$ is equivalent to the event {there is no particles of the process closer then $2r$ to the nucleus of the cell}.

Lemma 2 *Let R denote the radius of the minimal ball B_M centered in a nucleus and containing a Voronoi cell with respect to a Poisson process of intensity λ in $\mathbb{R}^2$. Then*

$$\begin{aligned} \mathbf{P}\{R > x\} &\le 1 - [1 - \exp(-\lambda x^2(\pi/4 + 2 - \sqrt{2}))]^8 \\ &\le 8\exp(-1.37\lambda x^2) \end{aligned} \tag{2}$$

Proof. First note that the event $\{R > x\}$ is equivalent to the event {there exists a point of the Voronoi cell at the distance at least x from 0 } which in its turn is equal to {there exists a circle of radius x containing the origin on its boundary and no particles of the process inside }.

Consider now a union of 8 circles of the radii x which centers have the polar coordinates of the form $(x, k\pi/4)$, $k = 0, 1, \ldots, 7$. Two of them are shown on figure 4. Then it is easy to see that any circle K of radius x having

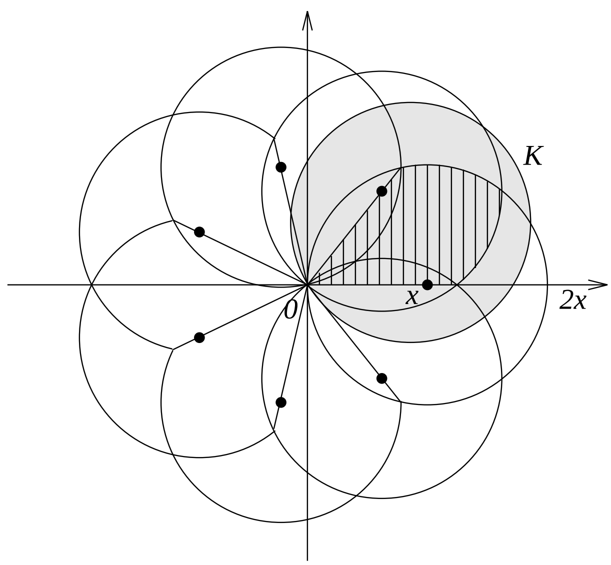

Figure 4

0 on its boundary contains at least one 'petal' formed by an intersection of a sector of a circle of the radius $\sqrt{2}x$ and the angle $\pi/4$ with two circles of radii x. One of eight such petals is shaded on figure 4. Thus $\mathbf{P}\{R > x\} <$ $\mathbf{P}$\{at least one of petals has no particle inside\}.

The estimate now is immediate since the area of a petal is equal to $(\pi/4 + 2 - \sqrt{2})x^2 \approx 1.37x^2$.

Theorem 2

$$-\log(1 + 4\frac{\lambda_{\mathcal{C}}}{\lambda_{\mathcal{S}}}) \le \liminf_{x\to\infty} \frac{\log \mathbf{P}\{N > x\}}{x}$$

$$\leq \limsup_{x\to\infty} \frac{\log \mathbf{P}\{N > x\}}{x} \leq -\log(1 + 0.44\frac{\lambda_{\mathcal{C}}}{\lambda_{\mathcal{S}}})$$

Proof. Denote by N_m and N_M the number of subscribers in B_m and B_M respectively for the Voronoi cell with nucleus 0. Then clearly $N_m < N < N_M$ a. s. Using that the number of subscribers in a fixed circle of radius r conforms to Poisson distribution with parameter $\lambda_{\mathcal{S}}\pi r^2$ we find that

$$\mathbf{E}\exp(zN_m) = \mathbf{E}\mathbf{E}\{\exp(zN_m)|\sigma(\mathcal{C}_0)\})$$

$$= \int_0^\infty \exp(\lambda_{\mathcal{S}}\pi z r^2(e^z - 1))\, d(1 - exp(-4\pi\lambda_{\mathcal{C}} r^2)) = \frac{4\lambda_{\mathcal{C}}}{4\lambda_{\mathcal{C}} + \lambda_{\mathcal{S}} - \lambda_{\mathcal{S}} e^z}$$

We have used here an explicit formula (1) for the distribution of the radius of B_m. Thus we obtained that the variable N_m has geometrical distribution with parameter $\frac{4\lambda_{\mathcal{C}}}{4\lambda_{\mathcal{C}}+\lambda_{\mathcal{S}}}$ with the tail

$$\mathbf{P}\{N_m > x\} \sim \exp(-x\log(1 + 4\frac{\lambda_{\mathcal{C}}}{\lambda_{\mathcal{S}}}))$$

Symbol $f(x) \sim g(x)$ here means that

$$\lim_{x\to\infty} \frac{\log f(x)}{\log g(x)} = 1$$

Similar arguments could be applied to estimate distribution of N_M. The situation here is slightly different since we do not know the exact distribution of the radius R of the ball B_R. However the estimate (2) implies that

$$\mathbf{E}\exp(zN_M) = \mathbf{E}\mathbf{E}\{\exp(zN_m)|\sigma(\mathcal{C})\})$$

$$\leq \int_0^\infty \exp(\lambda_{\mathcal{S}}\pi z r^2(e^z - 1))\, d(1 - exp(-1.37\lambda_{\mathcal{C}} r^2))^8$$

The last integral converges whenever $1.37\lambda_{\mathcal{C}} < \lambda_{\mathcal{S}}\pi z(e^z - 1)$ or $z < \log(1 + 0.44\frac{\lambda_{\mathcal{C}}}{\lambda_{\mathcal{S}}})$ that proves the theorem. □

Theorem 3 *There exist constants C_1 and C_2 depending on $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{S}}$ only such that*

$$C_1 \leq \liminf_{x\to\infty} \frac{\log \mathbf{P}\{l > x\}}{x^{2/3}} \leq \limsup_{x\to\infty} \frac{\log \mathbf{P}\{l > x\}}{x^{2/3}} \leq C_2$$

where $C_1 \geq -7.68\lambda_{\mathcal{C}}/\lambda_{\mathcal{S}}^{2/3}$ *and*

$$C_2 \leq -0.51\frac{\lambda_{\mathcal{C}}}{\lambda_{\mathcal{S}}^{2/3}}\left(1-0.74\sqrt{\frac{\lambda_{\mathcal{C}}}{\lambda_{\mathcal{S}}}}-0.55\frac{\lambda_{\mathcal{C}}}{\lambda_{\mathcal{S}}}\right)$$

provided that $\lambda_{\mathcal{C}}/\lambda_{\mathcal{S}} \leq 3.04$.

The proof of this theorem will be broken into several lemmas.

Let s_n denote n-th subscriber according to the distance from the commutator 0. Then

$$\mathbf{P}\{|s_n| \in (x, x+dx)\} = \frac{(\lambda_{\mathcal{S}}\pi x^2)^{n-1}}{(n-1)!}\exp(-\pi\lambda_{\mathcal{S}}x^2)\,2\lambda_{\mathcal{S}}\pi x dx + o(dx)$$

where $|\cdot|$ denotes Euclidean distance to 0. Thus the random variable s_n conforms to a generalized Gamma distribution $\Gamma_2(2n, \pi\lambda_{\mathcal{S}})$, where $\Gamma_\theta(\nu,\lambda)$ is given by the density

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ \theta\lambda^{\nu/\theta}x^{\nu-1}\exp(-\lambda x^\theta)\,/\Gamma(\nu/\theta) & \text{if } x \geq 0 \end{cases}$$

Later on we use the notation $\mu = \pi\lambda_{\mathcal{S}}$.

Mention the following property of the generalized Gamma distribution: if a random variable ξ has a Gamma distribution $\Gamma(\nu,\lambda) = \Gamma_1(\nu,\lambda)$, then $c^{-1}\xi^{1/\theta}$ has a $\Gamma_\theta(\nu\theta, \lambda c^\theta)$ distribution $(c>0, \theta>0)$. Thus taking a random variable η_n distributed as $\Gamma(n,\mu)$, we conclude that the variable $\sqrt{\eta_n}$ has the same $\Gamma_2(2n,\mu)$ distribution as $|s_n|$.

Lemma 3 *For all* $0 < \varepsilon < 1$

$$\mathbf{P}\{|\,|s_n| - \sqrt{n/\mu}\,| > \varepsilon\sqrt{n/\mu}\} < 2\exp(-Dn) \tag{3}$$

where $D = D(\varepsilon) = (3 - 2\log 2)\,\varepsilon^2 > 0$.

Proof. Let a random variable η_n has $\Gamma(n,\mu)$ distribution. Then for all $0 < \delta < 1$ Chebyshev's exponential inequality gives

$$\mathbf{P}\{\eta_n > n(\delta+1)/\mu\} \leq \inf_{0<\gamma<\mu} e^{-n\gamma(\delta+1)/\mu}\,\mathbf{E}e^{\gamma\eta_n}$$

$$= \inf_{0<\gamma<\mu} e^{-n\gamma(\delta+1)/\mu}\left(\frac{\mu}{\mu-\gamma}\right)^n = \exp(\,-n[\delta - \log(1+\delta)]\,) \tag{4}$$

for $\gamma = \delta\mu/(1+\delta)$.

Similarly

$$\mathbf{P}\{\eta_n < n(1-\delta)/\mu\} \leq \inf_{\gamma>0} e^{n\gamma(1-\delta)/\mu}\,\mathbf{E}e^{-\gamma\eta_n}$$

$$= \inf_{\gamma>0} e^{n\gamma(1-\delta)/\mu}\left(\frac{\mu}{\mu+\gamma}\right)^n = \exp(\,n[\delta + \log(1-\delta)]\,) \tag{5}$$

for $\gamma = \delta\mu/(1-\delta)$.

Now using the previous notations $\eta_n = |s_n|^2$ we have

$$\mathbf{P}\{|\,|s_n| - \sqrt{n/\mu}\,| > \varepsilon\sqrt{n/\mu}\} = \mathbf{P}\{|\sqrt{\eta_n} - \sqrt{n/\mu}\,| > \varepsilon\sqrt{n/\mu}\}$$

$$= \mathbf{P}\{\eta_n > (1+\varepsilon)^2 n/\mu\} + \mathbf{P}\{\eta_n < (1-\varepsilon)^2 n/\mu\}$$

Applying the estimates (4) and (5) for the last two terms we obtain the following upper bound:

$$\exp(-2n[\varepsilon + \varepsilon/2 - \log(1+\varepsilon)]) + \exp(-n[2\varepsilon - \varepsilon^2 - \log(1+2\varepsilon-\varepsilon^2)])$$

$$< 2\exp(-2n[\varepsilon + \varepsilon/2 - \log(1+\varepsilon)]) < 2\exp(-n(3-2\log 2)\varepsilon^2))$$

which proves the lemma. □

For a fixed configuration ω define $N(\varepsilon) = N(\varepsilon,\omega)$ as the minimal number satisfying $|\,|s_n| - \sqrt{n/\mu}\,| \le \varepsilon\sqrt{n/\mu}$ for all $n \ge N(\varepsilon)$. The estimate (3) and Borel-Cantelli lemma imply that the random variable $N(\varepsilon)$ is finite for almost all ω.

Lemma 4 *For all $u > 0$*

$$\mathbf{P}\{N(\varepsilon) > u\} < \frac{2\exp(-Du)}{1-\exp(-D)} \tag{6}$$

where D is the same as in Lemma 3

Proof. Using estimate (3) we have

$$\mathbf{P}\{N(\varepsilon) > u\} = \mathbf{P}\{\exists n > u \,:\, |\,|s_n| - \sqrt{n/\mu}\,| > \varepsilon\sqrt{n/\mu}\}$$

$$< \sum_{n=[u]+1}^{\infty} \mathbf{P}\{|\,|s_n| - \sqrt{n/\mu}\,| > \varepsilon\sqrt{n/\mu}\}$$

$$< \sum_{n=[u]+1}^{\infty} 2\exp(-Dn) \le \frac{2\exp(\,-D([u]+1)\,)}{1-\exp(-D)}$$

that gives (6). □

Lemma 5 *There exist positive constants A_2 and C_2 depending on $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{S}}$ only such that*

$$\mathbf{P}\{l > x\} < A_2\exp(-C_2 x^{2/3}) \tag{7}$$

for all x.

Proof. Remind that we denoted by R the radius of the minimal circle centered in 0 that contains the Voronoi cell $T(0)$ of the particle 0. It is clear that for all configurations ω

$$l = \sum_{s_n \in T(0)} |s_n| \le \sum_{n=1}^{\infty} |s_n| \mathbf{1}\{R \ge |s_n|\} \stackrel{\text{def}}{=} \Phi$$

Thus to get an upper bound on large deviations of l it suffices now to estimate the probability $\mathbf{P}\{\Phi > x\}$. The idea here is to divide the sum in the definition of Φ in two parts: up to the number N of lemma 4 which is finite for almost all configurations and the rest tail which behaves regularly.

We have the following sequence of estimates:

$$\Phi = \sum_{n=1}^{N} |s_n| \mathbf{1}\{R \ge |s_n|\} + \sum_{n=N+1}^{\infty} |s_n| \mathbf{1}\{R \ge |s_n|\}$$

$$\le N|s_N| + \sum_{n=N+1}^{\infty} \sqrt{n/\mu}\,(1+\varepsilon)\mathbf{1}\{R \ge \sqrt{n/\mu}\,(1-\varepsilon)\}$$

$$\le N^{3/2}(1+\varepsilon)/\sqrt{\mu} + (1+\varepsilon)/\sqrt{\mu} \sum_{n=1}^{[\mu R^2/(1-\varepsilon)^2]} \sqrt{n}$$

$$\le N^{3/2}\frac{1+\varepsilon}{\sqrt{\mu}} + \frac{2(1+\varepsilon)}{3\sqrt{\mu}} \left(\frac{\mu R^2}{(1-\varepsilon)^2} + 2\right)^{3/2}.$$

Now for any $0 \le \alpha \le 1$

$$\mathbf{P}\{\Phi > x\} \le \mathbf{P}\left\{N^{3/2}\frac{1+\varepsilon}{\sqrt{\mu}} > (1-\alpha)x\right\} + \mathbf{P}\left\{\frac{2(1+\varepsilon)}{3\sqrt{\mu}}\left(\frac{\mu R^2}{(1-\varepsilon)^2} + 2\right)^{3/2} > \alpha x\right\}$$

$$= \mathbf{P}\left\{N > \left(\frac{(1-\alpha)x\sqrt{\mu}}{1+\varepsilon}\right)^{2/3}\right\} + \mathbf{P}\left\{R > \frac{1-\varepsilon}{\sqrt{\mu}}\sqrt{\left(\frac{3\alpha x\sqrt{\mu}}{2(1+\varepsilon)}\right)^{2/3} - 2}\right\}$$

To estimate these terms we apply inequalities (6) and bound (2) of lemma 2 respectively that gives

$$\mathbf{P}\{\Phi > x\} < \frac{2}{1-\exp(D)} \exp\left(-D\left(\frac{(1-\alpha)\sqrt{\mu}}{1+\varepsilon}\right)^{2/3} x^{2/3}\right)$$

$$+ 8\exp\left(2.74\frac{\lambda_C}{\mu}\right)\exp\left(-1.37\lambda_C\frac{(1-\varepsilon)^2}{\mu}\left(\frac{3\alpha\sqrt{\mu}}{2(1+\varepsilon)}\right)^{2/3} x^{2/3}\right)$$

Actually the lemma is already proved. The only thing rest now is to minimize this sum by varying parameters ε and α.

Asymptotically the main contribution gives the heaviest tail. Therefore the best result could be obtained by equating the coefficients under the exponent:

$$D(1-\alpha)^{2/3} = 1.37\lambda_{\mathcal{C}}\frac{(1-\varepsilon)^2}{\mu}\left(\frac{3}{2}\right)^{2/3}\alpha^{2/3}$$

and hence $\alpha = (B+1)^{-1}$ where

$$B = \left(\frac{1.795\lambda_{\mathcal{C}}(1-\varepsilon)^2}{\mu D}\right)^{3/2}$$

Now as C_2 one can take the value

$$\max_{0<\varepsilon<1} 1.37\left(\frac{3}{2}\right)^{2/3}\frac{\lambda_{\mathcal{C}}(1-\varepsilon)^2}{(\mu(1+\varepsilon)(B+1))^{2/3}}$$

Note that in practice $\lambda_{\mathcal{C}}/\mu = \lambda_{\mathcal{C}}/\pi\lambda_{\mathcal{S}} << 1$. Provided that $\lambda_{\mathcal{C}}/\mu =\leq 0.97$ we then chose ε satisfying $D = 1.795\lambda_{\mathcal{C}}/\mu$ or $\varepsilon = 1.05\sqrt{\lambda_{\mathcal{C}}/\mu}$. This allows us to take as C_2 the following function:

$$1.13\frac{\lambda_{\mathcal{C}}}{\mu^{2/3}}\left(1 - 1.30\sqrt{\frac{\lambda_{\mathcal{C}}}{\mu}} + O\left(\frac{\lambda_{\mathcal{C}}}{\mu}\right)\right)$$

where $|O(\lambda_{\mathcal{C}}/\mu)| \leq 1.70\lambda_{\mathcal{C}}/\mu$.

Lemma 6 *There exists a positive constants A_1 and C_1 depending on $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{S}}$ only such that*

$$\mathbf{P}\{l > x\} > A_1\exp(-C_1x^{2/3}) \tag{8}$$

for all x.

Proof. Remind that we denoted by r the radius of the maximal circle with the center 0 contained in the Voronoi cell $T(0)$. Then for N as in lemma 4 and any $0 < \varepsilon < 1$

$$\begin{aligned}
l &= \sum_{s_n \in T(0)} |s_n| \geq \sum_{n=1}^{\infty} |s_n|\,\mathbf{1}\{r \geq |s_n|\} \geq \sum_{n=N+1}^{\infty} |s_n|\,\mathbf{1}\{r \geq |s_n|\} \\
&\geq \sum_{n=N+1}^{\infty} \sqrt{n/\mu}\,(1-\varepsilon)\,\mathbf{1}\{r > \sqrt{n/\mu}\,(1+\varepsilon)\} \\
&= \sum_{n=1}^{\infty} \sqrt{n/\mu}\,(1-\varepsilon)\,\mathbf{1}\{r > \sqrt{n/\mu}\,(1+\varepsilon)\} - \sum_{n=1}^{N} \sqrt{n/\mu}\,(1-\varepsilon) \\
&\geq \frac{2(1-\varepsilon)}{3\sqrt{\mu}}\left(\frac{\mu r^2}{(1+\varepsilon)^2} - 1\right)^{3/2} - \frac{2(1-\varepsilon)}{3\sqrt{\mu}}(N+1)^{3/2}
\end{aligned}$$

Now for any $0 < \alpha < 1$

$$\mathbf{P}\{l > x\} \geq \mathbf{P}\left\{\frac{2(1-\varepsilon)}{3\sqrt{\mu}}\left(\frac{\mu r^2}{(1+\varepsilon)^2} - 1\right)^{3/2} > (1+\alpha)x\right\}$$

$$-\mathbf{P}\left\{\frac{2(1-\varepsilon)}{3\sqrt{\mu}}(N+1)^{3/2} > \alpha x\right\}$$

$$= \mathbf{P}\left\{r > \frac{1+\varepsilon}{\sqrt{\mu}}\sqrt{\left(\frac{3\sqrt{\mu}(1+\alpha)x}{2(1-\varepsilon)}\right)^{2/3} + 1}\right\} - \mathbf{P}\left\{N > \left(\frac{3\sqrt{\mu}\,\alpha\, x}{2(1-\varepsilon)}\right)^{2/3} - 1\right\}$$

$$\geq e^{(1+\varepsilon)^2/\mu} \exp\left(-4 \cdot 1.5^{2/3}\pi\lambda_C(1+\varepsilon)^2\left(\frac{1+\alpha}{\mu(1-\varepsilon)}\right)^{2/3} x^{2/3}\right)$$

$$-\frac{4e}{1-e^{-D}}\exp\left(-1.5^{2/3}D\left(\frac{\alpha\sqrt{\mu}}{1-\varepsilon}\right)^{2/3} x^{2/3}\right)$$

To get the last inequality we have used lemma 1 for the first and estimate (6) for the second term. The statement of lemma now follows by minimizing this sum over ε and α. $\square$

Note here that the coefficient of $x^{2/3}$ under the first exponent is always greater then the one under the second exponent. Thus the main contribution gives the first exponent in the sense that

$$\liminf_{x\to\infty} \frac{\log \mathbf{P}\{l > x\}}{x^{2/3}} \geq -4 \cdot 1.5^{2/3}\pi\lambda_C(1+\varepsilon)^2\left(\frac{1+\alpha}{\mu(1-\varepsilon)}\right)^{2/3}$$

for all ε , α and thus

$$\liminf_{x\to\infty} \frac{\log \mathbf{P}\{l > x\}}{x^{2/3}} \geq -16.47\frac{\lambda_C}{\mu^{2/3}}$$

Combining all the above we get the statement of the theorem. $\square$

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